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Weighted Polynomial Approximation for Weights with Slowly Varying Extremal Density

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DEDICATED TO PROFESSOR M. R. OCCORSIO ON HIS 65TH BIRTHDAY AND TO MY TEACHER AND FRIEND LAJOS PINTÉR ON HIS 70TH BIRTHDAY

Polynomial approximation by weighted polynomials of the form $w^n(x)\, P_n(x)$ is investigated on closed subsets of the real line. It is known that the possibility of approximation is closely related to the density of an extremal measure associated with w via a weighted energy problem. It is also known that if in a neighborhood of a point x_0 this density is continuous and positive, then, in that neighborhood, any continuous function can be approximated. The aim of the present paper is twofold. On the one hand it is shown that the same approximation theorem is true if in a neighborhood of x_0 the density is slowly varying and is bounded away from 0. This allows singularities of logarithmic types. On the other hand, we also show that under some mild conditions, if the density at x_0 is slowly varying, then approximation is still possible even if the density vanishes at x_0. This is the first positive result for approximation with a vanishing density.

1. INTRODUCTION AND MAIN RESULTS

Recently a lot of attention has been devoted to weighted polynomial approximation with varying weights of the form $w^n P_n$, where the degree of P_n is at most n, i.e., in this approximation the weight varies together with the degree. Thus, P_n has to balance exponential oscillations in the weight, and this kind of approximation is much harder than ordinary weighted approximation.

This type of approximation has evolved from Lorentz' incomplete polynomials (where $w(x) = x^\alpha$ with some $\alpha > 0$ on $[0, 1]$), and appears in

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several different problems. For example, it appears in asymptotic properties of Freud type orthogonal polynomials (with $w(x)=e^{-|x|^\alpha}$ for some $\alpha>0$ on the real line) where it was a major tool in resolving G. Freud's conjectures by D. S. Lubinsky, E. A. Rahmanov, E. B. Saff, P. Nevai, A. Knopfmacher, and H. N. Mhaskar, and where it also plays a key role in proving strong asymptotics, as well (see the works [4, 5] by D. S. Lubinsky and E. B. Saff, and [9] by the author). It also plays an important role in multipoint Padé approximation where orthogonal polynomials with respect to varying weights appear in the error formula. Finally, we mention that it also has connection with generalizations of Wigner's semicircle law in statistical-mechanical models in statistical physics where it is related to finding the distribution of energy levels of quantum particles. For a general reference regarding this approximation, its history and its applications see the monographs [7, 9].

Let Σ be a closed subset of the real line, which, to avoid unnecessary technical complications, we assume to be regular with respect to the Dirichlet problem in $\mathbf{C}\setminus\Sigma$. Typically Σ consists of a finite number of intervals. Furthermore, let there be given a continuous and non-identically zero weight function w on Σ with the additional property that $|x|\,w(x)\to 0$ as $|x|\to\infty$ if Σ is unbounded. We shall call such weights *admissible*.

We are interested in approximating continuous functions on Σ by weighted expressions of the form w^nP_n, where P_n is a polynomial of degree at most n. There is a Stone–Weierstrass-type theorem for this kind of approximation (see [2]):

THEOREM A. *Let $\Sigma\subset\mathbf{R}$ be a closed set and w a continuous admissible weight on Σ. Then there exists a closed set $\mathscr{Z}(w)\subset\Sigma$ such that a continuous function f on Σ is the uniform limit of weighted polynomials w^nP_n, $n=1, 2, ...,$ if and only if f vanishes on $\mathscr{Z}(w)$.*

Thus, the problem of what functions can be approximated is equivalent to determining what points lie in $\mathscr{Z}(w)$. This latter problem is intimately related to the density of the following extremal measure: let us define the weighted energy integral of a Borel measure μ as

$$I_w(\mu):=\iint \log[\,|z-t|\,w(z)\,w(t)]^{-1}\,d\mu(z)\,d\mu(t),$$

and let us minimize $I_w(\mu)$ for all probability Borel measures μ with support in Σ. There is a unique measure $\mu=\mu_w$ minimizing the energy integral, this μ_w has compact support, and w is strictly positive on the support of μ_w. μ_w is called the *extremal* or *equilibrium measure associated with* w [7, Chap. I]. Now it turns out [9, Theorem 4.1] that all points outside the support supp(μ_w) of μ_w are bad points from the point of view of approximation, i.e., all such points belong to $\mathscr{Z}(w)$. On the other hand, we have [9, Theorem 4.2]

THEOREM B. *If μ_w has positive and continuous density (with respect to linear Lebesgue measure) in a neighborhood of x_0, then $x_0 \notin \mathscr{Z}(w)$.*

This theorem has two assumptions. The first one, the continuity of the extremal density, is not too restrictive; it is almost always satisfied in applications, and it is automatically guaranteed if in a neighborhood of the point x_0 lying in the interior of the support of μ_w the weight is $C^{1+\varepsilon}$-smooth for some $\varepsilon > 0$. In contrast, the second assumption, namely the strict positivity of the weight is very essential; up to now there exists no approximation theorem that applies to the case when the extremal density vanishes. The reason for this is that under normal circumstances a zero in the extremal density prevents approximation. In fact, it was shown by A. B. J. Kuijlaars [1] that if the density of μ_w is of the form $(1+o(1)) \times c\,|t-x_0|^\lambda$ in a neighborhood of x_0 with some $\lambda > 0$, then we have $x_0 \in \mathscr{Z}(w)$, i.e., a power type zero forces $x_0 \in \mathscr{Z}(w)$, which in turn forces all functions f, that are uniform limits of weighted polynomials $w^n P_n$, to vanish at x_0. We shall see that a "weaker than power type zero" allows approximation under rather general assumptions, thereby we shall obtain the first positive approximation result for vanishing extremal density.

What about a singularity in the extremal density? As an illustration consider the weights $w(x) = e^{-|x|^\alpha}$, $x \in \mathbf{R}$ with an $\alpha > 0$. In this case for $\alpha < 1$ the extremal density has a singularity of the type $(1+o(1))\,c\,|t|^{\alpha-1}$ around the origin, while for $\alpha = 1$ the singularity of the type $(1+o(1))\,c \log 1/|t|$ (for $\alpha > 1$ the extremal density is continuous). Now it turns out that the former, power type singularity is too strong: it was shown by A. B. J. Kuijlaars [1] that if the density of μ_w is of the form $(1+o(1))\,c\,|t-x_0|^\lambda$ in a neighborhood of x_0 with some $\lambda < 0$, then we have $x_0 \in \mathscr{Z}(w)$, i.e. a power type singularity implies $x_0 \in \mathscr{Z}(w)$, and hence in this case if f is uniformly approximable by weighted polynomials $w^n P_n$, then necessarily f vanishes at x_0. On the other hand, the logarithmic type singularity in the extremal density for the weight $w(x) = e^{-|x|}$, $x \in \mathbf{R}$, is too weak to prevent approximation, as was proven in [6]. It was shown by P. Simeonov [8] that a logarithmic type singularity of the form $c \log 1/|t-x_0| + o(1)$ in the density always allows approximation, i.e., in this case $x_0 \notin \mathscr{Z}(w)$.

The aim of this paper is twofold: we extend these results to slowly varying singularities in the extremal density and also show that for the slowly varying case the requirement that the density has a positive lower bound in a neighborhood of x_0 can be dropped under rather mild conditions.

To formulate the main theorems of the paper, let us introduce the following definition (cf. [10, Sect. 1.2]): we say that a positive function v defined in a neighborhood $[-a, a]$ of the origin is slowly varying at the origin, if v is continuous on $[-a, 0) \cup (0, a]$, $v(-t)/v(t) \to 1$ as $t \to 0$, and for every $r > 0$ we have $v(rt)/v(t) \to 1$ as $t \to 0+0$. v is called slowly varying at a point x_0 if the function $v^*(t) = v(t + x_0)$ is slowly varying at the origin.

If at a point x_0 the function v is continuous and positive, then clearly it is slowly varying. More typical examples are powers of absolute values of $\log |x|$, $\log |\log |x||$, etc., their products and positive linear combinations of such products. Faster growing or decreasing examples are the functions $\exp(\pm \log^\theta 1/|x|)$, $0 < \theta < 1$. Here the case $\theta = 1$ would mean a power type zero or singularity, which does not allow slow variance anymore. Our main theorems compared with the discussion after Theorem B show that there is a major difference between the $\theta < 1$ and $\theta = 1$ cases from the point of view of approximation. Note however, that a slowly varying function need not tend to zero or infinity, it may happen that

$$\liminf_{t \to x_0} v(t) = 0 \qquad \text{but} \qquad \limsup_{t \to x_0} v(t) = \infty. \tag{1.1}$$

It is also easy to see [10, Theorem 1.2.1] that if v is slowly varying at the origin, then uniformly for r lying in any interval $[\varepsilon, 1/\varepsilon]$, $\varepsilon > 0$ we have $v(rt)/v(t) \to 1$ as $t \to 0+0$.

One of our main results is

THEOREM 1.1. *Let us suppose that μ_w has a density $v_*(t)$ in a neighborhood D of x_0 such that $v_*(t)$ has a positive lower bound in D, it is continuous in D except possible at x_0, and is slowly varying at x_0. Then $x_0 \notin \mathscr{Z}(w)$.*

This theorem includes the case of continuous positive weights, but it is of real interest for weights for which $v_*(t)$ tends to infinity or slowly oscillates around the point x_0.

A. B. J. Kuijlaars [1] found a transformation with which results about inner points of the support can be transformed into results for endpoints of the support. Using the same procedure we can state the following consequence of Theorem 1.1.

THEOREM 1.2 *Let w be an admissible weight on Σ, and let $x_0 \in \operatorname{supp}(\mu_w)$ be a point such that for some $\delta > 0$ we have $\Sigma \cap (x_0 - \delta, x_0 + \delta) = [x_0, x_0 + \delta)$. Suppose that the extremal measure μ_w has density v_* in a right neighborhood of x_0 that satisfies there*

$$v_*(t) = |t - x_0|^{-1/2} v_{**}(t) \tag{1.2}$$

with a function $v_{**}(t)$ that has a positive lower bound in that neighborhood, is continuous in $(x_0, x_0 + \delta)$, and is slowly varying at x_0. Then $x_0 \notin \mathscr{Z}(w)$.

In this theorem the slow variation of $v_{**}(t)$ is required only from one side. Since the proof of this theorem requires the same transformation ($t \to u^2 + x_0$) as was used in [1], and with this transformation it reduces to Theorem 1.1, we shall only prove Theorem 1.1 (cf. also the discussion in [7, Sect. 6.1]).

In our second main theorem we drop the requirement that v has a positive lower bound around x_0 provided its behavior elsewhere is not too wild. This is the first positive result that ensures $x_0 \notin \mathscr{Z}(w)$ even if v may vanish at x_0. Note however, the theorem also applies to the case (1.1).

THEOREM 1.3. *Let the support of μ_w consist of finitely may intervals J_j, and suppose that $d\mu_w(x) = v_*(x)\,dx$ where v_* is a continuous function inside every J_j except for finitely many points, and v_* has only finitely many zeros. Assume further that if $A = \{a_i\}$ is the set consisting of the zeros and discontinuities of v and of the endpoints of the intervals J_j, then for each i either v_* is slowly varying at a_i, or there is a $\delta_i > -1$ such that $v(t) \sim |t - a_i|^{\delta_i}$ in a neighborhood of a_i. Under these conditions of x_0 is an inner point of the support of μ_w and v_* is slowly varying at x_0, then $x_0 \notin \mathscr{Z}(w)$.*

Here we used the notation $A \sim B$ to indicate that the ratio of the two sides lies in between two positive constants.

A similar statement holds when x_0 is an endpoint of one of the intervals J_j, see Theorem 1.2.

We do not know if the theorem holds without any restriction on v_* away from the point x_0. This problem can be reformulated as if in Theorem 1.1 the positive lower boundedness of v_* in a neighborhood of x_0 can be dropped. Nevertheless, the theorem is sufficiently general to cover all the interesting cases and is sufficiently convenient in applications.

In the next section we shall prove Theorem 1.1 based on several lemmas on slowly varying weights, the proof of which will be given in the last section of the paper. Theorem 1.3 will be proven in Section 3.

2. PROOF OF THEOREM 1.1

Let us suppose that in a neighborhood of the point x_0 the extremal measure μ_w has a density that is bounded from below by a positive constant, continuous in that neighborhood with the possible exception of the point x_0, and is slowly varying at x_0. We have to prove that under these assumptions $x_0 \notin \mathscr{Z}(w)$, and to this end it is enough to show that some function f with $f(x_0) \neq 0$ is the uniform limit on Σ of weighted polynomials $w^n P_n$. Without loss of generality we assume that $x_0 = 0$.

Let $v_*(t)$ be the density of μ_w in a neighborhood of 0, say in $[-r, r]$, and let $2d_* > 0$ be a lower bound for v_* in that neighborhood. Let us select and fix an $0 < \alpha < r$, an integer m and a continuous function $h_*(t) \leqslant v_*(t)$ on $[-\alpha, \alpha]$ in such a way that $d_* \leqslant h_*(t)$, and we have

$$\int_0^\alpha (v_*(t) - h_*(t))\, dt = \frac{1}{2m}, \qquad \int_{-\alpha}^0 (v_*(t) - h_*(t))\, dt = \frac{1}{2m}. \tag{2.1}$$

This is always possible. For example, start with selecting $h_1(t) = d_*$ for $t \in [-\alpha, \alpha]$, where a small α is selected so that one of the equalities in (2.1) (with h_* replaced by h_1) holds with some integer $m \geqslant 2$, and the integral in the other equality is $\geqslant 1/2m$. Suppose, for example, that we have equality over the interval $[0, \alpha]$. The infimum of

$$\int_{-\alpha}^0 (v_*(t) - h_2(t))\, dt$$

for all continuous functions $h_2(t)$ on $[-\alpha, \alpha]$ subject to the conditions $v_*(t) \geqslant h_2(t) \geqslant d^*$ for $t \in [-\alpha, 0]$, $h_2(t) = d_*$ for $t \in [0, \alpha]$, is clearly zero (recall that v^* is continuous away from the origin), therefore there is such an h_2 for which the preceding expression is smaller than $1/2m$. Now an appropriate convex linear combination of h_1 and h_2 is suitable for h_*.

We set $v(t) = m(v_*(t) - h_*(t))$ if $t \in [-\alpha, \alpha]$ and $v(t) = 0$ otherwise, and let $w_1(x) = \exp(U^v(x))$, where

$$U^v(x) = \int \log \frac{1}{|x-t|}\, v(t)\, dt$$

is the logarithmic potential associated with v. We also set

$$w_2(x) = (w(x)/w_1(x)^{1/m})^{m/(m-1)}.$$

If we define $d\mu_2(x) = (m/(m-1))\, d\mu_w(x)$ for $x \notin [-\alpha, \alpha]$ and $d\mu_2(x) = (mh_*(x)/(m-1))\, dx$ for $x \in [-\alpha, \alpha]$, then the potential $U^{\mu_2}(x)$ equals

$$\frac{m}{m-1}\left(U^{\mu_w}(x) - \frac{U^v(x)}{m}\right),$$

and since $\operatorname{supp}(\mu_2) = \operatorname{supp}(\mu_w)$, it follows from the defining properties of equilibrium measures that here μ_2 is the equilibrium measure for w_2, i.e. $\mu_2 = \mu_{w_2}$. In fact, the extremal measure μ_w in the weighted energy problem has the characterization (see [7, Theorems I.1.3 and I.3.3]) that with some constant F_w we have

$$U^{\mu_w}(x) = \log w(x) + F_w \qquad \text{for} \quad x \in \operatorname{supp}(\mu_w), \tag{2.2}$$

and

$$U^{\mu_w}(x) \geqslant \log w(x) + F_w \qquad \text{for} \quad x \in \Sigma \backslash \operatorname{supp}(\mu_w). \tag{2.3}$$

Now if we use this for w and μ_w, then we can see that the same relations hold true for w_2 and μ_2 and this proves $\mu_{w_2} = \mu_2$.

Hence, μ_{w_2} has the positive and continuous density $(m/(m-1))\, h_*(t)$ on $(-\alpha, \alpha)$, so by Theorems A and B any continuous function that vanishes outside this interval is a uniform limit of weighted polynomials $w_2^n R_n$ with $\deg(R_n) \leqslant n$.

Let f be such a continuous function which does not vanish at the origin, but vanishes outside $(-\alpha/2, \alpha/2)$. It is enough to show that this f can be uniformly approximated by weighted polynomials $w^n P_n$. According to what we have just established, for every n there are polynomials $R_{4(m-1)n}$ of degree at most $4(m-1)\, n$ such that

$$w_2^{4(m-1)\, n}(x)\, R_{4(m-1)\, n}(x) \to f(x), \qquad \text{as} \quad n \to \infty \tag{2.4}$$

uniformly in $x \in \Sigma$. Now we invoke the following theorem to be proven below.

THEOREM 2.1. *For every n there are polynomials S_{4n} of degree at most $4n$ such that $w_1^{4n}(x)\, S_{4n}(x) \to 1$ as $n \to \infty$ uniformly on compact subsets of $(-\alpha, \alpha)$, and such that the functions $\{w_1^{4n}(x)\, |S_{4n}(x)|\}_n$ are uniformly bounded on the real line; more precisely*

$$\lim_{n \to \infty} \|w_1^{4n} S_{4n}\|_\infty = 1. \tag{2.5}$$

Then $w_1^{4n} w_2^{4(m-1)\, n} S_{4n} R_{4(m-1)\, n}$ uniformly tends to f on Σ, for the weighted polynomials $w_2^{4(m-1)\, m} R_{4(m-1)\, n}$ uniformly tend to zero outside the interval $[-\alpha/2, \alpha/2]$, and the weighted polynomials $w_1^{4n} S_{4n}$ are uniformly bounded there. Furthermore, here the degree of $S_{4n} R_{4(m-1)\, n}$ is at most $4mn$, while $w_1^{4n} w_2^{4(m-1)\, n} = w^{4mn}$ by the choice of the function w_2. This proves the existence of weighted polynomials $w^k P_k$ converging to f for the sequence of degrees $\{k = 4mn\}_{n=1}^{\infty}$. To convert this argument into one which covers all degrees k, all we have to do is to choose, instead of (2.4), for each $s = 0, 1, ..., 4m-1$ polynomials $R_{4(m-1)\, n,\, s}$ of degree at most $4(m-1)\, n$ in such a way that

$$w_2^{4(m-1)\, n}(x)\, R_{4(m-1)\, n,\, s}(x) \to f(x)/w^s(x)$$

uniformly on Σ (recall that f vanishes outside $[-\alpha/2, \alpha/2]$ and w is positive on $[-\alpha, \alpha]$ because it is positive on the support of μ_w). Then the products $w_1^{4n} w_2^{4(m-1)\, n} S_{4n} R_{4(m-1)\, n,\, s}$ uniformly tend to f/w^s as $n \to \infty$, and so by setting $P_k = S_{4n} R_{4(m-1)n,\,s}$ for $k = 4mn + s$, $n = 1, 2, ...,$ $s = 0, 1, ...,$ $4m - 1$ we have $w^k P_k \to f$ uniformly, and the proof is complete pending the proof of Theorem 2.1.

Proof of Theorem 2.1. It is clear from the definition of $v(t)$ and from the assumption of the theorem that $v(t)$ is slowly varying at the origin. Furthermore, $v(t)$ has equal integrals 1/2 on $(-\alpha, 0)$ and on $(0, \alpha)$ by its definition. Without loss of generality we may assume that $\alpha = 1$, for otherwise we can consider the density $v^*(t) = \alpha v(\alpha t)$. Since we have $U^v(\alpha x) = U^{v^*}(x) + \log 1/\alpha$, and this constant can be incorporated into the polynomials, the statement of Theorem 2.1 for v and α and for v^* and $\alpha = 1$ are equivalent.

It is enough to prove that for every n there are polynomials Q^*_{2n} of degree at most $2n$ such that $w_1^{2n}(x)\,|Q^*_{2n}(x)| \to 1$ uniformly on compact subsets of $(-1, 1)$, and such that the supremum norm on the real line of the function $w_1^{2n}(x)\,|Q^*_{2n}(x)|$ tends to 1 as $n \to \infty$, for then we can set $S_{4n}(x) = |Q^*_{2n}(x)|^2$ (which is again a polynomial).

We divide the interval $[-1, 1]$ into $2n$ subintervals

$$I_{j,n} = I_j, \qquad j = -n, ..., -1, 1, ..., n$$

by the points $x_{j,n} = x_j$ for which

$$\int_0^{x_{j,n}} v(t)\,dt = \frac{j}{2n}, \qquad j = -n, ..., n.$$

In particular, $x_0 = 0$, $x_{-n} = -1$ and $x_n = 1$. If we set $I_{j,n} = [x_{j-1,n}, x_{j,n}]$ for $j = 1, ..., n$ and $I_{j,n} = [x_{j,n}, x_{j+1,n}]$ for $j = -n, ..., -1$, then $\bigcup_j I_{j,n} = [-1, 1]$, and $\{I_{j,n}\}_j$ is a partition of $[-1, 1]$ into subintervals with the property

$$\int_{I_{j,n}} v(t)\,dt = \frac{1}{2n}. \tag{2.6}$$

We fix a number $0 < a < 1$ arbitrarily. Let

$$\xi_{j,n} = 2n \int_{I_{j,n}} t v(t)\,dt \tag{2.7}$$

be the weight point of v on $I_{j,n}$, and with some large, but fixed positive integer $L \geq 4$ we define the polynomial

$$Q_{2n}(x) = \prod_{j \neq 0} (x - \xi_{j,n} + iL\,|I_{j,n}|)$$

of degree $2n$. We claim that an appropriate constant multiple of these polynomials satisfy the requirements provided we shall let $L \to \infty$ very slowly compared to n.

Since

$$w_1^{2n}(x)\,|Q_{2n}(x)| = e^{2nU^v(x)}\,|Q_{2n}(x)| = \exp(2nU^v(x) + \log |Q_{2n}(x)|),$$

and here

$$2nU^v(x) + \log |Q_{2n}(x)|$$

$$= \sum_{j \neq 0} 2n \int_{I_{j,n}} (\log |x - \xi_{j,n} + iL\,|I_{j,n}|| - \log |x - t|)\, v(t)\, dt,$$

we have to estimate

$$\sum_{j \neq 0} 2n \int_{I_{j,n}} \log \left| \frac{x - \xi_{j,n} + iL\,|I_{j,n}|}{x - t} \right| v(t)\, dt, \tag{2.8}$$

which is the difference $\Sigma_1 - \Sigma_2$ of

$$\Sigma_1 := \sum_{j \neq 0} 2n \int_{I_{j,n}} \log \left| \frac{x - t + iL\,|I_{j,n}|}{x - t} \right| v(t)\, dt$$

and

$$\Sigma_2 := \sum_{j \neq 0} 2n \int_{I_{j,n}} \log \left| \frac{x - t + iL\,|I_{j,n}|}{x - \xi_{j,n} + iL\,|I_{j,n}|} \right| v(t)\, dt,$$

and we shall separately consider these two sums. We are going to prove that for $x \in [-a, a]$ (where $0 < a < 1$ is the number that was fixed above) we have $\Sigma_1 = (1 + o(1))\, c_L + O(L^{-1/2})$, while for all $x \in \mathbf{R}$ the estimates $\Sigma_1 \leqslant (1 + o(1))\, c_L + O(L^{-1/2})$ and $\Sigma_2 = O(L^{-1/2})$ are true as $n \to \infty$, where c_L is a constant depending only on L and $c_L \to \infty$ as $L \to \infty$. These relations should be understood as $n \to \infty$; more precisely they hold in the following sense. For every fixed $L \geqslant 4$ and $0 < a < 1$ the $o(1)$ terms tend to 0 as $n \to \infty$, and their convergence to 0 is uniform in $x \in [-a, a]$ or $x \in \mathbf{R}$, respectively. The constants in the $O(L^{-1/2})$ terms are independent of L (and $x \in [-a, a]$ or $x \in \mathbf{R}$, respectively), provided n is sufficiently large; in other words there is an absolute constant C, and for all fixed $L \geqslant 4$ and $0 < a < 1$ there is a number $n_{a,L}$ such that if $n \geqslant n_{a,L}$, then the absolute value of the $O(L^{-1/2})$ terms are less than $CL^{-1/2}$ independently of $x \in [-a, a]$ or $x \in \mathbf{R}$, respectively. Now these estimates show that if $L = L_n$ tends very slowly to ∞ compared to n, then for the polynomials $Q_{2n}^*(x) = e^{-c_{L_n}} Q_{2n}(x)$ the weighted expression $\exp(2nU^v(x))\,|Q_{2n}^*(x)|$ uniformly converges to 1 on every closed subinterval of $(-1, 1)$, and at the same time

$$\lim_{n\to\infty} \|\exp(2nU^v)\, Q_{2n}^*\|_\infty = 1$$

holds, which clearly implies (2.5) for the polynomial $S_{4n} = |Q_{2n}^*(x)|^2$. This completes the proof of Theorem 2.1. ∎

For later use let us mention the following. Let $\varepsilon > 0$, and $x \in \mathbf{R} \setminus [-1-\varepsilon, 1+\varepsilon]$. If n is large enough, then in the estimate of Σ_1 below the terms in Σ_{31} are missing, and this Σ_{31} accounts for the constant c_L above. Therefore, uniformly for $x \notin [-1-\varepsilon, 1+\varepsilon]$ we have $\Sigma_1 = O(L^{-1/2})$ and $\Sigma_2 = O(L^{-1/2})$. Furthermore $c_L \to \infty$ as $L \to \infty$ (see (2.11) below), therefore the weighted expression $e^{2nU^v(x)}\, |Q_{2n}^*(x)|$ tends to zero uniformly outside $[-1-\varepsilon, 1+\varepsilon]$. We record this fact as

$$e^{2nU^v(x)}\, |Q_{2n}^*(x)| \to 0 \tag{2.9}$$

uniformly outside $[-1-\varepsilon, 1+\varepsilon]$ (provided $L = L_n \to \infty$ very slowly compared to $n \to \infty$).

2.1. *Estimate of* Σ_1

It is shown in Lemma 7 in Section 4 below that, as $n \to \infty$, uniformly in $x \in \mathbf{R}$ and j the jth term in Σ_1 is $(1 + o(1))$ times

$$\frac{1}{|I_{j,n}|} \int_{I_{j,n}} \log \left| \frac{x - t + iL\, |I_{j,n}|}{x - t} \right| dt,$$

therefore it is enough to estimate the sum of these terms, i.e. to estimate

$$\Sigma_3 := \sum_{j \neq 0} \frac{1}{|I_{j,n}|} \int_{I_{j,n}} \log \left| \frac{x - t + iL\, |I_{j,n}|}{x - t} \right| dt.$$

Without loss of generality we may assume $x \geq 0$. Let $x \in I_{j_0}$, and let us break the preceding sum for summation over $|j - j_0| \leq L^3$ and the rest, i.e.,

$$\Sigma_3 = \sum_{|j - j_0| \leq L^3} + \sum_{|j - j_0| > L^3} = \Sigma_{31} + \Sigma_{32}.$$

Let $x = x_{j_0 - 1, n} + y\, |I_{j_0, n}|$, $0 \leq y \leq 1$, and let us also write the variable $t \in I_{j,n}$, $|j - j_0| \leq L^3$ in the form $t = x_{j-1,n} + \tau\, |I_{j,n}|$, $0 \leq \tau \leq 1$ (if $I_{j,n}$ happens to lie in $[-1, 0]$, then this has to be modified to $t = x_{j,n} + \tau\, |I_{j,n}|$, but for simplicity we keep the preceding notation for such $I_{j,n}$'s, as well). On making use of the substitution $t \to \tau$ we obtain

$$\frac{1}{|I_{j,n}|}\int_{I_{j,n}} \log\left|\frac{x-t+iL\,|I_{j,n}|}{x-t}\right| dt$$

$$= \int_0^1 \log\left|\frac{(x_{j_0-1,n}-x_{j-1,N})/|I_{j,n}|+y-\tau+iL+y(|I_{j_0,n}|/|I_{j,n}|-1)}{(x_{j_0-1,n}-x_{j-1,n})/|I_{j,n}|+y-\tau+y(|I_{j_0,n}|/|I_{j,n}|-1)}\right| d\tau$$

$$= \int_{a_{n,j}(y)}^{a_{n,j}(y)+1} \log\left|\frac{y-t+iL}{y-t}\right| dt, \tag{2.10}$$

where

$$a_{n,j}(y) = -(x_{j_0-1,n}-x_{j-1,n})/|I_{j,n}| - y(|I_{j_0,n}|/|I_{j,n}|-1).$$

For each j with $|j-j_0| \leqslant L^3$ we have by Lemma 5 that $|I_{j,n}|/|I_{j_0,n}| \to 1$ as $n \to \infty$, and the same lemma also implies that then

$$(x_{j_0-1,n}-x_{j-1,n})/|I_{j,n}| \to j_0 - j,$$

which means that each $a_{n,j}(y)$ tends to $j-j_0$ uniformly in y as $n \to \infty$. Therefore, the last integral in (2.10) tends to

$$\int_{j-j_0}^{j-j_0+1} \log\left|\frac{y-t+iL}{y-t}\right| dt,$$

and all together we obtain that as $n \to \infty$

$$\Sigma_{31} = (1+o(1)) \sum_{|j-j_0| \leqslant L^3} \int_{j-j_0}^{j-j_0+1} \log\left|\frac{y-t+iL}{y-t}\right| dt$$

$$= (1+o(1)) \int_{-L^3}^{L^3+1} \log\left|\frac{y-t+iL}{y-t}\right| dt.$$

Since for $y \in [0, 1]$ and $|t| \geqslant L^3$ we have

$$\log\left|\frac{y-t+iL}{y-t}\right| = \frac{1}{2}\log\left(1+\frac{L^2}{(y-t)^2}\right) \leqslant \frac{L^2}{(y-t)^2},$$

and the integral of the right hand side for $t \in (-\infty, -L^3) \cup (L^3+1, \infty)$ is at most $2/L$, we finally conclude that

$$\Sigma_{31} = (1+o(1)) \int_{-\infty}^{\infty} \log\left|\frac{y-t+iL}{y-t}\right| dt + O\left(\frac{1}{L}\right)$$

$$= (1+o(1))\, c_L + O\left(\frac{1}{L}\right),$$

because the last integral is actually independent of y. Simple contour integral transformation or integration by parts shows that $c_L = \pi L$, but we shall only use that

$$c_L = \frac{1}{2}\int_{-\infty}^{\infty} \log \frac{t^2+L^2}{t^2}\, dt \to \infty \tag{2.11}$$

as $L \to \infty$, which is perfectly clear. Note also, that here the constant in the error term $O(1/L)$ is independent of L, in fact, this error term is less than $3/L$ for sufficiently large n.

Next we estimate Σ_{32}. For $x \in I_{j_0, n}$, $t \in I_{j, n}$ and $|j - j_0| > L^3$ we have

$$\log \left| \frac{x-t+iL\,|I_{j,n}|}{x-t} \right| = \frac{1}{2} \log\left(1 + \frac{L^2\,|I_{j,n}|^2}{(x-t)^2}\right) \leqslant \frac{2L^2\,|I_{j,n}|^2}{(x_{j_0,n}-x_{j,n})^2}$$

(see Lemma 5 which easily implies that $|x-t| \geqslant |x_{j_0,n} - x_{j,n}|/2$ if n is sufficiently large), and by Lemma 6 here

$$(x_{j_0,n} - x_{j,n})^2 \geqslant c\,|j_0 - j|^{11/6}\,|I_{j,n}|^2 \tag{2.12}$$

with some positive constant c independent of j, j_0, and n. Hence

$$\Sigma_{32} \leqslant C \sum_{|j-j_0| \geqslant L^3} \frac{L^2}{|j-j_0|^{11/6}} \leqslant \frac{C}{L^{1/2}},$$

and the constant C here is independent of L.

Summarizing our estimates in this subsection we can state that for $x \in [-a, a]$ and sufficiently large n, say $n \geqslant n_{a,L}$, we have

$$\Sigma_1 = (1+o(1))\, c_L + O(L^{-1/2}), \tag{2.13}$$

where the constant in O is independent of L.

If we glance at this proof for other $x \in [-1, 1]$ we see that actually (2.13) holds for all x around which L^3 intervals on both sides belong to $[-1, 1]$ (i.e., for which $n - L^3 > j_0 > -n + L^3$). When this condition is not satisfied, then the sum in Σ_{31} does not contain all the terms, so we can only say that then

$$0 \leqslant \Sigma_1 \leqslant (1+o(1))\, c_L + C(L^{-1/2}), \tag{2.14}$$

is true with some constant C independent of $x \in [-1, 1]$ and L provided n is sufficiently large.

Finally, if $x \notin [-1, 1]$, then (2.14) follows from its validity for $x \in [-1, 1]$. In fact, for any $t \in [-1, 1]$ the function

$$\log \left| \frac{x-t+iL\,|I_{j,n}|}{x-t} \right|$$

is a positive function of x which is decreasing on the interval $[1, \infty)$, and is increasing on the interval $(-\infty, -1]$, therefore, the validity of (2.14) for $x = \pm 1$ implies the same inequality for $x \in (-\infty, -1] \cup [1, \infty)$.

Thus, (2.14) is uniformly true for all $x \in \mathbf{R}$ for all large n with the C and $o(1)$ independent of L.

2.2. *Estimate of* Σ_2

We write

$$\log \left| \frac{x - t + iL\, |I_{j,n}|}{x - \xi_{j,n} + iL\, |I_{j,n}|} \right| = \log \left| 1 + \frac{\xi_{j,n} - t}{x - \xi_{j,n} + iL\, |I_{j,n}|} \right|,$$

and note that for $t \in I_{j,n}$ here the last ratio is necessarily smaller than $1/L$ in absolute value. Therefore

$$\begin{aligned} \log & \left| \frac{x - t + iL\, |I_{j,n}|}{x - \xi_{j,n} + iL\, |I_{j,n}|} \right| \\ &= \Re \log \left(1 + \frac{\xi_{j,n} - t}{x - \xi_{j,n} + iL\, |I_{j,n}|} \right) \\ &= \Re \frac{\xi_{j,n} - t}{x - \xi_{j,n} + iL\, |I_{j,n}|} + O\left(\frac{(\xi_{j,n} - t)^2}{(x - \xi_{j,n})^2 + L^2\, |I_{j,n}|^2} \right). \end{aligned}$$

The real part on the right hand side equals

$$(\xi_{j,n} - t)\, \Re \frac{1}{x - \xi_{j,n} + iL\, |I_{j,n}|} = (\xi_{j,n} - t)\, c(x, j),$$

where $c(x, j)$ is independent of t. But, by the definition of the weight point $\xi_{j,n}$, the integral of this real part against $v(t)$ over $I_{j,n}$ is zero:

$$\int_{I_{j,n}} (\xi_{j,n} - t)\, c(x, j)\, v(t)\, dt = 0.$$

Therefore, using also (2.6) it follows that

$$\begin{aligned} 2n \int_{I_{j,n}} \log \left| \frac{x - t + iL\, |I_{j,n}|}{x - \xi_{j,n} + iL\, |I_{j,n}|} \right| v(t)\, dt &= O\left(\frac{(\xi_{j,n} - t)^2}{(x - \xi_{j,n})^2 + L^2\, |I_{j,n}|^2} \right) \\ &= O\left(\frac{|I_{j,n}|^2}{(x_{j_0,n} - x_{j,n})^2 + L^2\, |I_{j,n}|^2} \right) \end{aligned}$$

where the constant in O is independent of x, j and n. Now the sum of these terms for $|j - j_0| \leqslant L$ is at most $C(2L+1)/L^2 \leqslant C/L$ (where the two C's on the two sides are not necessarily the same), while for the sum with $|j-j_0|>L$ we obtain from Lemma 6 the bound

$$C\sum_{|j-j_0|>L}\frac{1}{|j-j_0|^{3/2}}\leqslant\frac{C}{L^{1/2}}.$$

These estimates show that $\Sigma_2\leqslant C/L^{1/2}$, and this is what we needed to prove.

Note that this proof works for all $x\in\mathbf{R}$ (see also the proof of (2.14) for all $x\in\mathbf{R}$ in the preceding subsection).

3. PROOF OF THEOREM 1.3

Let us suppose that in a neighborhood of the point x_0 the extremal measure μ_w has a density that is continuous in that neighborhood with the possible exception of the point x_0, and is slowly varying at x_0, and otherwise this density is continuous and positive with a finite number of exceptions where it has a power type or slowly varying behavior. Note that we do not assume now any lower bound on the density around the point x_0. We have to prove that under these assumptions $x_0\notin\mathscr{Z}(w)$, and to this end it is enough to show that some function f with $f(x_0)\neq 0$ is the uniform limit on Σ of weighted polynomials w^nP_n. Without loss of generality we assume that $x_0=0$.

Let $v_*(t)$ be the density of μ_w. Let us select and fix a small $0<\alpha$, so that in $[-\alpha,\alpha]$ the function v_* is continuous with the only possible exception of the point 0. Then choose a function $0\leqslant h_*(t)\leqslant v_*(t)$ on the interior of the support $\mathscr{S}_w$ of μ_w with the following properties:

- $h_*(t)=v_*(t)$ if $t\notin[-\alpha,\alpha]$,
- $h^*(t)=0$ if $t\in[-\alpha/2,\alpha/2]$,
- $0<h_*(t)<v_*(t)$ if $t\in[-\alpha,-\alpha/2]\cup[\alpha/2,\alpha]$,
- $h_*(t)$ is linear in a left, resp. in a right neighborhood of $-\alpha/2$ and $\alpha/2$, respectively,
- $h_*(t)$ is continuous on $[-\alpha,\alpha]$, and

$$\int_0^\alpha (v_*(t)-h_*(t))\,dt=\frac{1}{2m},\qquad \int_{-\alpha}^0 (v_*(t)-h_*(t))\,dt=\frac{1}{2m},$$

where $m\geqslant 2$ is some positive integer. Using the slow variation of v_* around 0, and by selecting α sufficiently close to the origin, it is not difficult to see with the method of Section 2 that it is always possible to select such an h_*.

As in Section 2 we set $v(t)=m(v_*(t)-h_*(t))$ if $t\in[-\alpha,\alpha]$ and $v(t)=0$ otherwise, and let $w_1(x)=\exp(U^v(x))$,

$$w_2(x)=(w(x)/w_1(x)^{1/m})^{m/(m-1)}. \tag{3.1}$$

Exactly as in Section 2 the equilibrium measure for w_1 has density $m(v_*-h_*)$, while the equilibrium measure for w_2 has density $(m/(m-1))\,h_*$.

We are going to prove

THEOREM 3.1. *Let $-\alpha\leqslant\beta<\gamma\leqslant\alpha$ be fixed. Then for every n there are nonnegative polynomials $S_{4n}^{\beta,\gamma}$ of degree at most $4n$ such that $w_1^{4n}(x)\,S_{4n}^{\beta,\gamma}(x)\to 1$ as $n\to\infty$ uniformly on compact subsets of (β,γ), $w_1^{4n}(x)\,S_{4n}^{\beta,\gamma}(x)\to 0$ uniformly on every set $\mathbf{R}\setminus[\beta-\varepsilon,\gamma+\varepsilon]$, $\varepsilon>0$, and such that the functions $\{w_1^{4n}(x)\,|S_{4n}^{\beta,\gamma}(x)|\}_n$ are uniformly bounded on the real line with a bound independent of $-\alpha\leqslant\beta,\gamma\leqslant\alpha$.*

Let f be an arbitrary nonnegative continuous function with compact support in $(-\alpha/2,\alpha/2)$. It is easy to see that if M is given, then there are intervals $[\beta_k,\gamma_k]$, $1\leqslant k\leqslant N$ in $[-\alpha,\alpha]$ such that all the numbers $\{\beta_k,\gamma_k\}_{k=1}^N$ are distinct, and if χ_k denotes the characteristic function of $[\beta_k,\gamma_k]$, then

$$\left|f(x)-\frac{1}{N}\sum_{k=1}^{N}\chi_k(x)\right|\leqslant\frac{1}{M}.$$

Now if

$$S_{4n}(x)=\frac{1}{N}\sum_{k=1}^{N}S_{4n}^{\beta_k,\gamma_k}(x),$$

then from the listed properties of the polynomials $S_{4n}^{\beta,\gamma}$ it follows that $w_1^{4n}S_{4n}$ is close to the function f;

$$\limsup_{n\to\infty}\|f-w_1^{4n}S_{4n}\|_\infty\leqslant\frac{C}{M},$$

where C is independent of M (depends only on the bound for the weighted polynomials in Theorem 3.1). Therefore, by letting $M\to\infty$ very slowly compared to n, we get a sequence of polynomials S_{4n} for which $w_1^{4n}S_{4n}$ uniformly converges to f on the real line.

This result immediately leads to its own generalization: the same thing can be achieved by polynomials S_{4n-i_n} of degree at most $4n-i_n$, where $\{i_n\}$ is a sequence converging to infinity,

$$\lim_{n\to\infty}\|f-w_1^{4n}S_{4n-i_n}\|_\infty=0. \tag{3.2}$$

In fact, all we have to do is to approximate f/w_1^s as above,

$$\lim_{n\to\infty} \|f/w_1^s - w_1^{4n} S_{4n}\|_\infty = 0,$$

which, after multiplication through by w_1^s yields

$$\lim_{n\to\infty} \|f - w_1^{4n+s} S_{4n}\|_\infty = 0. \tag{3.3}$$

Now this implies that there is a sequence $s_n \to \infty$ of integers such that together with (3.3)

$$\lim_{n\to\infty} \|f - w_1^{4n+s_n} S_{4n}\|_\infty = 0$$

also holds, and this is (3.2) in a different form.

Next we use

THEOREM 3.2. *Let w_2 be any weight function on Σ such that its equilibrium measure has a density that satisfies the hypothesis of Theorem* 1.3, *and such that $(-\alpha/2, \alpha/2)$ is disjoint from the support of μ_{w_2}, but*

$$U^{\mu_{w_2}}(x) = \log w_2(x) + F_{w_2} \tag{3.4}$$

holds on the interval $(-\alpha/2, \alpha/2)$ (cf. (2.2)). Let further $i_n \to \infty$ arbitrarily. Then there is a sequence $R_{4(m-1)n+i_n}$ of polynomials of corresponding degree $4(m-1)n + i_n$, $n = 1, 2, ...,$ for which $w_2^{4(m-1)n} R_{4(m-1)n+i_n}$ are uniformly bounded on compact subsets of Σ, and

$$w_2^{4(m-1)n} R_{4(m-1)n+i_n}(x) \to 1$$

uniformly on compact subsets of $(-\alpha/2, \alpha/2)$.

We can apply this theorem to the w_2 defined in (3.1), because for this weight the extremal measure has density $(m/(m-1))\, h_*$ that satisfies the hypothesis of Theorem 3.2 by the assumptions in Theorem 1.3 and by the construction of the function h_* (in fact, this construction was done so as to facilitate the assumptions in Theorem 3.2), and (3.4) also holds because the same inequality was true for μ_w and μ_{w_1} (note that in both cases the interval $[-\alpha/2, \alpha/2]$ belongs to the support of μ_w and μ_{w_1}).

Now the product weighted polynomials

$$\begin{aligned} w_1^{4n} w_2^{4(m-1)n} S_{4n-i_n} R_{4(m-1)n+i_n} &= w^{4mn} S_{4n-i_n} R_{4(m-1)n+i_n} \\ &=: w^{4mn} P_{4mn} \end{aligned}$$

with S_{4n-i_n} from (3.2) uniformly tend to f on compact subset of Σ, for the weighted polynomials $w_1^{4n} S_{4n-i_n}$ uniformly tend to zero outside the support of f (lying in the interval $[-\alpha/2, \alpha/2]$), and the weighted polynomials $w_2^{4(m-1)m} R_{4(m-1)n+i_n}$ are uniformly bounded there on compact subsets, while on the support itself we have convergence to f and 1, respectively.

From here one can go to a full sequence $w^n P_n$ satisfying the requirements locally uniformly exactly as in Section 2.

Finally, we have to show that local uniform convergence implies uniform convergence for $w^n P_n$. If Σ is compact, then we are done. If Σ is unbounded, then let I be an interval containing the support of the extremal measure μ_w such that outside I we have

$$U^{\mu_w}(x) \geqslant \log w(x) + F_w + 1. \tag{3.5}$$

Since we have assumed that $|x| w(x) \to 0$ as $|x| \to \infty$, there is such an I. It is well known (see, for example, [7, Theorem III.2.1]) that

$$|P_n(x)| \leqslant \|w^n P_n\|_I \exp(n(-U^{\mu_w}(x) + F_w))$$

for all $x \in \mathbf{R}$. Now multiplying this with $w^n(x)$ and making use of (3.5) we can conclude that if $\{w^n P_n\}$ is uniformly bounded on I, then it uniformly tends to 0 outside I, and with this the proof of Theorem 1.3 is complete pending the proofs of Theorems 3.1 and 3.2.

Thus, to complete the proof of Theorem 1.3 we have to verify these two theorems.

3.1. *Proof of Theorem* 3.1

Here we follow the construction in the proof of Theorem 2.1, scale $[-\alpha, \alpha]$ to $[-1, 1]$, and denote the image of the density $m(v_* - h_*)$ under this scaling by v. Let $w_1(x) = \exp(U^v(x))$, where U^v denotes the potential of the measure with density v.

Consider the polynomials

$$Q_{2n}(x) = \prod_{j \neq 0} (x - \xi_{j,n} + iL\, |I_{j,n}|)$$

constructed in that proof. We have seen that $\exp(2nU^v(x))\, |Q_{2n}(x)|\, e^{-cL_n}$ is uniformly bounded on the real line, it converges to 1 uniformly on compact subsets of $(-1, 1)$, and uniformly converges to 0 outside every interval $[-1-\varepsilon, 1/+\varepsilon]$ (cf. (2.9)).

Now let $-1 \leqslant \beta < \gamma \leqslant 1$ be arbitrary. We prove the theorem by moving the zeros $\xi_{j,n} - iL/|I_{j,n}|$ with $\xi_{j,n} \notin [\beta, \gamma]$ to the real axis, that is by replacing the factors $(x - \xi_{j,n} + iL/|I_{j,n}|)$ in Q_{2n} with $\xi_{j,n} \notin [\beta, \gamma]$ by $(x - \xi_{j,n})$.

Let $\tilde{Q}_{2n}$ be the polynomial that we obtain this way. Note that this operation cannot increase the absolute value of the polynomial at any point of the real line, i.e., for $x \in \mathbf{R}$ we have $|\tilde{Q}_{2n}(x)| \leqslant |Q_{2n}(x)|$, thus the uniform boundedness of $w_1^{2n} |\tilde{Q}_{2n}|$ on $\mathbf{R}$ follows from that of $w_1^{2n} |Q_{2n}|$ (recall that Q_{2n} has degree $2n$).

For any $x \in [-1, 1]$ the above change of roots introduces the factor

$$\prod_{\xi_{j,n} \notin [\beta, \gamma]} \frac{|x - \xi_{j,n}|^2}{|x - \xi_{j,n}|^2 + L^2 |I_{j,n}|^2} \tag{3.6}$$

in the square of the absolute value of the polynomial, and on the intervals $[-1, \beta - \varepsilon]$ and $[\gamma + \varepsilon, 1]$ this is at most $1/L^2$ just by looking at the factor for which $x \in I_{j,n}$. Since $L = L_n \to \infty$, this proves that the polynomials converge uniformly to 0 on any interval $[-1, \beta - \varepsilon]$ and $[\gamma + \varepsilon, 1]$. Next we show that this implies uniform convergence to 0 on $(-\infty, -1]$ and on $[1, \infty)$, as well. In fact, on using (2.8) we see that the logarithm of $w_1^{2n}(x) |\tilde{Q}_{2n}(x)|$ is a sum of the terms

$$2n \int_{I_{j,n}} \log \left| \frac{x - \xi_{j,n} + il_j}{x - t} \right| v(t)\, dt,$$

where $l_j = L |I_{j,n}|$ for $\xi_{j,n} \in [\beta, \gamma]$ and $l_j = 0$ if $\xi_{j,n} \notin [\beta, \gamma]$. Thus, it is enough to verify that each of these terms is nonnegative, and monotone decreasing on $[1, \infty)$ and monotone increasing on $(-\infty, -1]$. The nonnegativity has to be checked only for $l_j = 0$ and then it is an immediate consequence of Jensen's inequality for convex functions, since $-\log |x - t|$ is a convex function of $t \in (-1, 1)$ for $x \notin (-1, 1)$, and $\xi_{j,n}$ was chosen to be the weight point of v on $I_{j,n}$. On taking derivative, the monotonicity on, say, $[1, \infty)$ amounts the same as

$$2n \int_{I_{j,n}} \left(\frac{x - \xi_{j,n}}{(x - \xi_{j,n})^2 + l_j^2} - \frac{1}{x - t} \right) v(t)\, dt \leqslant 0.$$

It is enough to verify this for $l_j = 0$, in which case it follows by another application of Jensen's inequality, because $1/(x - t)$ is a convex function of $t \in (-1, 1)$ for any x lying in $[1, \infty)$.

Finally, if $x \in [\beta + \varepsilon, \gamma - \varepsilon]$, then the reciprocal of the factor in (3.6) is at most

$$\prod_{\xi_{j,n} \notin [\beta, \gamma]} \left(1 + \frac{L^2 |I_{j,n}|^2}{|x - \xi_{j,n}|^2} \right),$$

the logarithm of which can be estimated from above as

$$L^2 \max_j |I_{j,n}| \sum_{\xi_{j,n} \notin [\beta, \gamma]} \frac{|I_{j,n}|}{|x-\xi_{j,n}|^2}.$$

Here the factor in front of the summation tends to zero as $n \to \infty$ (recall that $L = L_n \to \infty$ very slowly, so we may assume that $L^2 \max_j |I_{j,n}| \to 0$). The sum itself converges to

$$\int_{t \in [-1, \beta] \cup [\gamma, 1]} \frac{1}{(x-t)^2} \, dt \leqslant \frac{2}{\varepsilon},$$

and we obtain that the factor (3.6) is $(1+o(1))$ on compact subsets of (β, γ). Therefore, together $w_1^{2n} |Q_{2n}|$, also the function $w_1^{2n} |\tilde{Q}_{2n}|$ converges to 1 uniformly on compact subsets of (β, γ), and the proof is complete.

3.2. *Proof of Theorem* 3.2.

In this proof we closely follow the proof of [7, Theorem VI.4.2]. There the following result was proved:

THEOREM C. *Let μ be a measure of total mass* 1 *and suppose that* supp(μ) *is an interval J, and $d\mu(x) = v(x)\, dx$ where v is a continuous function inside J except for finitely many points, and v has only finitely many zeros. Assume further that if $A = \{a_i\}$ is the set consisting of the zeros and discontinuities of v and of the endpoints of the interval J, then for each i there is a $\delta_i > -1$ such that $v(t) \sim |t - a_i|^{\delta_i}$ in a neighborhood of a_i. Then there are polynomials*

$$P_n(x) = \prod_{j=0}^{n-1} (x - \xi_j)$$

with all their zeros in the support of μ such that, for some constant C,

$$|P_n(x)| \leqslant C \exp(-n U^{\mu}(x))$$

for all $x \in \mathbf{R}$.

The relation $v(t) \sim |t - a_i|^{\delta_i}$ is assumed only from one side if a_i is an endpoint of a subinterval of supp(μ).

It is not difficult to see that the proof of Theorem C remains valid if, instead of power type singularities and zeros, we also allow slow variance of v at certain points of A; we shall not go into details regarding this fact, just mention that a slowly varying function behaves more regularly than a function of type $v(t) \sim |t - a_i|^{\delta_i}$, and it is at least as easy to handle as the latter one.

We shall, however, need an extension of Theorem C to the effect that the polynomials constructed in the proof of [7, Theorem VI.4.2] also satisfy

$$\exp(-nU^{\mu}(x))\,|P_n(x)| \to 1 \tag{3.7}$$

uniformly outside every neighborhood of J. To see that we have to recall how the polynomials P_n were constructed in [7]. In fact, this construction has much in common with what we have used in this paper, and it runs as follows.

Let n be an even number (when n is odd, use $n-1$ in place of n below, and add appropriately one more zero to get exact degree n). Partition $J=:[a,b]$ by the points $a=t_0<t_1<\cdots<t_n=b$ into n intervals I_j, $j=0,1,\ldots,n-1$ with $\mu(I_j)=1/n$, and let ξ_j be the weight point of the restriction of μ to I_j; i.e.,

$$\xi_j = n\int_{I_j} tv(t)\,dt. \tag{3.8}$$

Set

$$P_n(t) = \prod_{j=0}^{n-1}(t-\xi_j).$$

It was shown in [7, Theorem VI.4.2] that these P_n satisfy the requirements of Theorem C. Now we indicate why (3.7) is true. For $x\notin J$ write

$$-\log|P_n(x)| - nU^{\mu}(x) = \sum_{j=0}^{n-1} n\int_{I_j}\log\left|\frac{x-t}{x-\xi_j}\right| v(t)\,dt =: \sum_{j=0}^{n-1} L_j(x). \tag{3.9}$$

Since the function $\log|x-t|$ is concave on I_j, we have by Jensen's inequality

$$n\int_{I_j}\log|x-t|\,v(t)\,dt \leqslant \log|x-\xi_j|,$$

and hence every term $L_j(x)$ in (3.9) is at most 0. This proves that

$$\exp(-nU^{\mu}(x))\,|P_n(x)| \leqslant 1 \tag{3.10}$$

for all $x\notin J$.

We can write the integrand in $L_j(x)$ as

$$\log\left|1+\frac{\xi_j-t}{x-\xi_j}\right| = \frac{\xi_j-t}{x-\xi_j} + O\left(\left|\frac{\xi_j-t}{x-\xi_j}\right|^2\right).$$

Now utilizing the weight point property of ξ_j it follows that

$$L_j(x) = n \int_{I_j} O\left(\left|\frac{\xi_j - t}{x - \xi_j}\right|^2\right) v(t)\, dt$$
$$= O\left(\frac{|I_j|^2}{(x - \xi_{j0})^2}\right), \tag{3.11}$$

because the integrals

$$\int_{I_j} \frac{|\xi_j - t|}{x - \xi_j} v(t)\, dt$$

vanish. Thus, as $n \to \infty$

$$\sum_j L_j(x) \leqslant \max_j |I_j| \sum_j \frac{|I_j|}{(x - \xi_j)^2} \to 0$$

uniformly outside any neighborhood of J, because the factor in front of the sum tends to zero, and the sum itself converges to the integral

$$\int_J \frac{1}{(x-t)^2}\, dt.$$

These prove the claim concerning (3.7). Note that up to now the degree of the polynomial P_n matched the exponent in $\exp(nU^\mu(x))$.

Now suppose that the support of μ consists of the intervals $J_1, ..., J_k$. Set $\alpha_j = \mu(J_i)$, and choose numbers n_j such that their sum is n, and for every $j = 1, ..., k$ we have $|n_j - [\alpha_j n]| \leqslant 1$. Since the sum of the α_j's is $\|\mu\| = 1$, this is clearly possible. Now we can apply the preceding consideration to every measure $\mu_j = (1/\alpha_j)\, \mu|_{J_j}$ and to the degrees n_j to get polynomials P_{n_j} of degree n_j with all their zeros lying in J_j that satisfy Theorem C on J_j with μ etc. replaced by μ_j etc. We can write with the product polynomial

$$P_n(x) = \prod_{j=0}^{k} P_{n_j}(x)$$

of degree n the product $\exp(nU^\mu)\, P_n$ in the form

$$\exp(nU^\mu(x))\, P_n(x) = \prod_j \exp(n_j U^{\mu_j}(x))\, P_{n_j}(x) \exp(U^{\nu_n}(x)),$$

where

$$\nu_n = n\mu - \sum_j n_j \mu_j = n \sum_j \alpha_j \mu_j - \sum_j n_j \mu_j = \sum_j (n\alpha_j - n_j)\, \mu_j.$$

Therefore, by Theorem C and (3.7) for the measures μ_j it follows that

$$\exp(nU^{\mu}(x))\,|P_n(x)|\,\exp(-U^{\nu_n}(x))$$

are uniformly bounded on the real line, and tend to 1 on compact subsets of the complement of the support of the measure μ. In particular, this holds for all degrees $4(m-1)\,n$ instead of n. Since here the degree of $P_{4(m-1)\,n}$ is at most $4(m-1)\,n$, while in Theorem 3.2 we are allowed degree $4(m-1)\,n+i_n$ where i_n tends to infinity, we can use the excess degree to get rid of the factor $\exp(-U^{\nu_n}(x))$. In fact, it is easy to see that the measures ν_n are from the set

$$\sum \tau_j\mu_j, \qquad -1\leqslant \tau_j \leqslant 1 \quad \text{for all } j,$$

and the potentials of the measures in this set are uniformly equicontinuous on every bounded subinterval of the real line (because the potentials of each μ_j are continuous everywhere). Thus, $\exp(U^{\nu_n})$ is a uniformly equicontinuous set of functions on every finite subinterval of $\mathbf{R}$, and as such can be uniformly approximated by polynomials, i.e. there are polynomials T_{i_n} of degree at most i_n such that

$$T_{i_n}(x)\,\exp(U^{\nu_{4(m-1)n}}(x)) \to 1$$

uniformly on every finite subinterval of $\mathbf{R}$ (in fact, this is an immediate consequence of Jackson's theorem on polynomial approximation). Thus, for the choice $R_{4(m-1)n+i_n}(x)=P_{4(m-1)n}(x)\,T_{i_n}(x)$, all the properties set forth in Theorem 3.2 are satisfied.

4. LEMMAS ON SLOWLY VARYING FUNCTIONS

In what follows we shall prove several lemmas that are used in the proof of the main theorem of the paper. In this section v denotes a weight function on $[-1, 1]$ with total integral 1 that is continuous and positive on $[-1, 0)\cup(0, 1]$ and slowly varying around 0. We also assume that

$$\int_0^1 v=\int_{-1}^0 v=\tfrac{1}{2}.$$

Note that we *do not assume* that v has a positive lower bound on $[-1, 1]$, i.e., v may have a zero at the origin.

We recall the definition of the points $x_j = x_{j,n}$; they are the unique points in $[-1, 1]$ for which

$$\int_0^{x_{j,n}} v(t)\,dt = \frac{j}{2n}, \qquad j = -n, ..., n. \tag{4.1}$$

In particular, $x_{0,n} = 0$, $x_{-n,n} = -1$, $x_{n,n} = 1$, and if we set $I_j = I_{j,n} = [x_{j-1,n}, x_{j,n}]$ for $j = 1, ..., n$ and $I_j = I_{j,n} = [x_{j,n}, x_{j+1,n}]$ for $j = -n, ..., -1$, then $\bigcup_j I_{j,n} = [-1, 1]$, and $\{I_{j,n}\}_j$ is a partition of $[-1, 1]$ into subintervals with the property

$$\int_{I_{j,n}} v(t)\,dt = \frac{1}{2n}.$$

The interval $I_{0,n}$ is not defined, and to have unified formulae we set $I_{0,n} = I_{1,n}$.

We shall use the notation $F \sim G$ if the ratio F/G lies between two positive constants independently of the parameters and variables in the range for which $F \sim G$ is indicated.

We recall that

$$v(rt)/v(t) \to 1 \qquad \text{as} \quad t \to 0 \pm 0 \tag{4.2}$$

uniformly for r lying in any compact subinterval of $(0, \infty)$ [10, Theorem 1.2.1].

LEMMA 1. *For every $\tau > 0$ there is a C such that for $x, y \in [-1, 1]$, $|y| \leqslant |x|$*

$$C\left(\frac{|y|}{|x|}\right)^{-\tau} \geqslant \frac{v(y)}{v(x)} \geqslant \frac{1}{C}\left(\frac{|y|}{|x|}\right)^{-\tau}. \tag{4.3}$$

In particular, for every $\tau > 0$ there is a C such that

$$v(x) \leqslant C\,|x|^{-\tau} \qquad \textit{for all} \quad x \in [-1, 1]. \tag{4.4}$$

Proof. Since $v(x) \sim v(-x)$, without loss of generality we may assume that $0 \leqslant y \leqslant x$. For $x \geqslant y \geqslant x/2$ the inequalities follow from (4.2), thus let $0 < y < x/2$. Let $0 < \varepsilon < 1$ be arbitrary, and choose to this ε a $\delta > 0$ so that for $0 < s/2 \leqslant t < s \leqslant \delta$ we have $v(s) \leqslant (1 + \varepsilon)\,v(t)$. This gives for $x \leqslant \delta$ the inequalities

$$v(2y) \leqslant (1+\varepsilon)\, v(y)$$
$$v(4y) \leqslant (1+\varepsilon)\, v(2y)$$
$$\vdots$$
$$v(2^m y) \leqslant (1+\varepsilon)\, v(2^{m-1} y)$$
$$v(x) \leqslant (1+\varepsilon)\, v(2^m y),$$

where m is chosen so that $x/2 < 2^m y \leqslant x$. On multiplying these inequalities together we obtain

$$v(x) \leqslant (1+\varepsilon)^{m+1}\, v(y) \leqslant 2(|x|/|y|)^{\tau}\, v(y)$$

with $\tau = \log_2(1+\varepsilon)$. Since here $0 < \varepsilon < 1$ is arbitrary, this proves the lower estimate in (4.3) for $x \leqslant \delta$.

Finally, when $x > \delta$, then the proof is the same, just for $2^j y > \delta$ we have to use the inequality $v(2^{j+1} y) \leqslant C v(2^j y)$ valid with some constant C, because on the interval $[\delta, 1]$ the function v is positive and continuous (note that there can be at most $\log_2 1/\delta$ such j).

This proves the lower estimate in (4.3). The upper one can be similarly proven.

Inequality (4.4) with x replaced by y follows if we set $x = 1$ into (4.3). ∎

Lemma 2. *For every $\tau > 0$ there is a $C > 0$ such that for $\eta < 1$*

$$\int_0^{\eta x} v(u)\, du \leqslant C \eta^{1-\tau} \int_0^x v(u)\, du \qquad \textit{for all} \quad x \in [0, 1], \tag{4.5}$$

and a similar estimate is true for $x \in [-1, 0]$. More generally, if $I \subset [0, x]$ is an arbitrarily interval of length ηx, then

$$\int_I v(u)\, du \leqslant C \eta^{1-\tau} \int_0^x v(u)\, du \qquad \textit{for all} \quad x \in [0, 1], \tag{4.6}$$

and a similar estimate is true for negative x.

Proof. Inequality (4.5) can be obtained by making use of the substitution $u = \eta t$ on the left-hand side, and making use of (4.3). Let now $I = [\delta, \delta + \eta x]$. If $\delta \leqslant \eta x$, then the integral on the left of (4.6) is at most as large as the integral over $[0, 2\eta x]$, hence in this case (4.6) follows from (4.5). On the other hand, if $\delta > \eta x$, then $v(t) \sim v(u)$ for $t \in I$ and $u \in [\delta, 2\delta]$, hence

$$\int_I v(u)\, du \leqslant C \frac{\eta x}{\delta} \int_\delta^{2\delta} v(u)\, du,$$

furthermore by (4.5) we have

$$\int_{\delta}^{2\delta} v(u)\,du \leqslant \int_{0}^{2\delta} v(u)\,du \leqslant C\left(\frac{2\delta}{x}\right)^{1-\tau} \int_{0}^{x} v(u)\,du.$$

Now (4.6) is a consequence of these formulae since we are considering the case $\delta > \eta x$. ∎

Lemma 3. *We have*

$$\int_0^x v(t)\,dt = (1+o(1))\,xv(x) \qquad \textit{as} \quad x \to 0, \tag{4.7}$$

and

$$\int_0^x v(t)\,dt \sim xv(x) \qquad \textit{for} \quad x \in [-1, 1]. \tag{4.8}$$

As a consequence, for any $\varepsilon > 0$

$$\begin{aligned}\int_0^{yx} v(t)\,dt &= (1+o(1))\,yxv(yx)\\ &= (1+o(1))\,yxv(x) \qquad \textit{as} \quad x \to 0\end{aligned} \tag{4.9}$$

uniformly in $\varepsilon \leqslant y \leqslant 1/\varepsilon$.

Proof. We may again assume $x > 0$. Given $\varepsilon > 0$ we can choose by (4.5) an $\varepsilon > \eta > 0$ such that we have

$$\int_0^{\eta x} v(u)\,du \leqslant \varepsilon \int_0^x v(u)\,du$$

for all $x \in [0, 1]$. Then there is a $\delta > 0$ such that for $0 < x < \delta$ the inequality $v(t)/v(u) \leqslant 1 + \varepsilon$ is true for all $0 < x < \delta$ and $\eta x < t$, $u < x$. Thus,

$$\int_0^x v(t)\,dt \leqslant \frac{1}{1-\varepsilon} \int_{\eta x}^x v(t)\,dt \leqslant \frac{1+\varepsilon}{1-\varepsilon}\, v(x)(x - \eta x) \leqslant \frac{1+\varepsilon}{1-\varepsilon}\, xv(x),$$

and

$$\int_0^x v(t)\,dt \geqslant \int_{\eta x}^x v(t)\,dt \geqslant \frac{1}{1+\varepsilon}\, v(x)(x - vx) \geqslant \frac{1-\varepsilon}{1+\varepsilon}\, xv(x),$$

and this proves formula (4.7) in the lemma.

Relations (4.8)–(4.9) are immediate consequences, if we also use that v is a slowly varying weight. ∎

Return now to the points $x_j = x_{j,n}$ introduced in (4.1). In the following proofs we shall frequently write x_j instead of $x_{j,n}$. Since for any fixed $j \geqslant 1$

$$\int_0^{x_{j,n}} v(t)\,dt = \frac{j}{2n} = j \int_0^{x_{1,n}} v(t)\,dt,$$

it immediately follows from (4.9) that $x_{j,n} = (1+o(1))\, jx_{1,n}$. In a similar fashion can one prove that $x_{-j,n} = (1+o(1))\, jx_{-1,n}$. Since $v(-x)/v(x) \to 1$ as $x \to 0$, we also have $x_{-1,n} = -(1+o(1))\, x_{1,n}$, hence we can conclude that for all fixed j (positive or negative)

$$x_{j,n} = (1+o(1))\, jx_{1,n}. \tag{4.10}$$

Note that (4.8) also implies that

$$x_{j,n} \sim \frac{j}{nv(x_{j,n})} \qquad \text{for all } x_{j,n}, \quad j \neq 0. \tag{4.11}$$

Lemma 4. *For every $0 < \tau < 1$ there is a C such that for all j and k, $|k| \leqslant |j|$ we have*

$$\frac{1}{C}\left(\frac{|j|}{|k|}\right)^{1-\tau} \leqslant \frac{|x_{j,n}|}{|x_{k,n}|} \leqslant C\left(\frac{|j|}{|k|}\right)^{1+\tau}. \tag{4.12}$$

Proof. Assume again that j and k are positive. We have by (4.11) $x_j \sim j/nv(x_j)$, and $x_k \sim k/nv(x_k)$. Therefore,

$$\frac{x_j}{x_k} \sim \frac{j}{k}\,\frac{v(x_k)}{v(x_j)},$$

and if we apply (4.3) to the last fraction, then after rearrangement we obtain

$$\frac{1}{C_1}\left(\frac{j}{k}\right)^{1/(1+\tau)} \leqslant \frac{x_j}{x_k} \leqslant C_1\left(\frac{j}{k}\right)^{1/(1-\tau)},$$

and this is equivalent to what we needed to prove. ∎

Lemma 5. *As $n \to \infty$*

$$\frac{|I_{j+1,n}|}{|I_{j,n}|} \to 1 \tag{4.13}$$

uniformly in all the indices j.

Proof. Since $|I_{j,n}|=x_{j,n}-x_{j-1,n}$ for positive j and $|I_{j,n}|=x_{j+1,n}-x_{j,n}$ for $j\leqslant 0$, for $j=0, \pm 1, \pm 2$ the statement follows from (4.10). Therefore, we may assume without loss of generality that $j\geqslant 2$ (for negative j the proof is similar). From (4.12) it follows that for $j\geqslant 2$ we have $x_{j,n}<Cx_{j-1,n}$ with some C, and hence from the slow variation of v we obtain that for every $\varepsilon>0$ there is a $\delta>0$ such that if $x_{1,n}<x_{j,n}\leqslant\delta$, then

$$\frac{1}{2n}=\int_{I_{j,n}} v(t)\,dt \overset{1+\varepsilon}{\sim} |I_{j,n}|\,v(x_{j,n}),$$

where $F\overset{A}{\sim}G$ means that $1/A\leqslant F/G\leqslant A$.

We apply this for j and $j+1$, and note that if δ is sufficiently small, then, by the slow variation of v and by $x_{j+1,n}<Cx_{j,n}$ established above, we have $v(x_{j,n})\overset{1+\varepsilon}{\sim}v(x_{j+1,n})$. From these we obtain for $x_{j,n}\in[x_{2,n},\delta]$

$$|I_{j+1,n}| \overset{(1+\varepsilon)^3}{\sim} |I_{j,n}|. \tag{4.14}$$

Finally, on the interval $[\delta, 1]$ the function v is uniformly continuous and positive, from which (4.14) immediately follows for those indices for which $x_{j,n}$ lies in this interval, provided n is sufficiently large. ∎

Note that the proof also gives

$$|I_{j,n}|=(1+o(1))\,\frac{1}{2nv(x_{j,n})} \tag{4.15}$$

uniformly in j, and also that for the weight point $\xi_{j,n}$ of v on $I_{j,n}$ defined as

$$\xi_{j,n}=2n\int_{I_{j,n}} tv(t)\,dt \tag{4.16}$$

we have

$$v(x_{j-1,n})=(1+o(1))\,v(x_{j,n})=(1+o(1))\,v(\xi_{j,n}) \tag{4.17}$$

for $j>1$,

$$\xi_{j,n}=(1+o(1))\,\frac{x_{j-1,n}+x_{j,n}}{2} \tag{4.18}$$

for $j\geqslant 1$,

$$v(x_{j,n})=(1+o(1))\,v(x_{j+1,n})=(1+o(1))\,v(\xi_{j,n}) \tag{4.19}$$

for $j < -1$, and

$$\xi_{j,n} = (1+o(1))\,\frac{x_{j,n}+x_{j+1,n}}{2}, \tag{4.20}$$

for $j \leqslant -1$, and the $o(1)$ in these relations is uniform in the indices j indicated. Note that we have excluded $j=1$ in (4.17) and $j=-1$ in (4.19), for v may have a singularity at the origin, nevertheless (4.18) and (4.20) are true also in these cases.

In fact, (4.17) follows from ideas applied in the preceding proof (use the slow variation of v on the interval $I_{j,n}$ for $j \neq \pm 1$) and so only (4.18) needs clarification and only for $j=1$, in which case it can be verified from (4.5) and (4.7) and the slow variation of v, with arguments applied before.

LEMMA 6. *For every* $0<\tau<1$ *there is a* $c>0$ *such that*

$$|x_{j,n}-x_{k,n}| \geqslant c\,|j-k|^{1-\tau}\,|I_{j,n}| \qquad \textit{for all } j, k, \textit{and } n. \tag{4.21}$$

Proof. First let j and k be positive.

If $j/2 \leqslant k \leqslant 2j$, then we have by (4.12) $1/C \leqslant x_j/x_k \leqslant C$ with some C independent of n, and then we get for example for $k<j$

$$\begin{aligned}|x_j - x_k| = x_j - x_k = \sum_{k<s\leqslant j} |I_s| \sim \sum_{k<s\leqslant j} \frac{1}{nv(x_s)}\\ \sim (j-k)\,\frac{1}{nv(x_j)} \sim (j-k)\,|I_j|,\end{aligned}$$

where we used (4.15) and the slow variation of v.

If $k<j/2$, then

$$|x_j - x_k| = x_j - x_k \sim x_j \sim j/nv(x_j) \sim j\,|I_j| \sim (j-k)\,|I_j|,$$

where we used (4.11) and (4.15). If, however, $2j<k$, then

$$\begin{aligned}|x_j - x_k| = x_k - x_j \sim x_k \geqslant c(k/j)^{1-\tau}\,x_j\\ \sim (k/j)^{1-\tau}\,j/nv(x_j) \geqslant ck^{1-\tau}\,|I_j|\\ \sim (k-j)^{1-\tau}\,|I_j|,\end{aligned}$$

where we used Lemma 4, (4.11) and (4.15).

Similar proof works if both j and k are negative. Finally, let, say, k be negative and j positive. It follows from Lemma 4 that there is a $C>1$ such that if $|k| > Cj$, then $|x_k| > 2x_j$. Thus, if $|k| > Cj$, then we can deduce from the already proven cases

$$\begin{aligned}|x_j - x_k| \sim |x_k| \sim |x_{-k}| \sim |x_{-k} - x_j| \\ \geqslant c \, ||k| - j|^{1-\tau} \, |I_j| \sim |k-j|^{1-\tau} \, |I_j|.\end{aligned}$$

If, however, $|k| \leqslant Cj$, then by Lemma 4

$$|x_j - x_k| = x_j + |x_k| \sim x_j \sim j/nv(x_j) \sim j \, |I_j| \sim (j-k) \, |I_j|. \quad \blacksquare$$

LEMMA 7. *Let $L \geqslant 4$ be any fixed positive number. Then with the weight points $\xi_{j,n}$ of v on $I_{j,n}$ we have*

$$\begin{aligned}&2n \int_{I_{j,n}} \log \left| \frac{x - t + iL \, |I_{j,n}|}{x-t} \right| v(t) \, dt \\ &\quad = (1+o(1)) \, 2nv(\xi_{j,n}) \int_{I_{j,n}} \log \left| \frac{x - t + iL \, |I_{j,n}|}{x-t} \right| dt \\ &\quad = (1+o(1)) \frac{1}{|I_{j,n}|} \int_{I_{j,n}} \log \left| \frac{x - t + iL \, |I_{j,n}|}{x-t} \right| dt \qquad (4.22)\end{aligned}$$

as $n \to \infty$, and this relation uniformly holds in j and $x \in \mathbf{R}$.

Proof. The last relation is a consequence of (4.15) and (4.17)–(4.19) (for $j = \pm 1$ use also (4.18) and (4.20) and the slow variation of v).

The first asymptotics is immediate for $j \neq \pm 1$, since in that case $v(t) = (1+o(1)) \, v(u)$ for $t, u \in I_j$ by the slow variance of v (cf. also Lemma 4).

Therefore, it is left to prove the first relation only for $j = \pm 1$. Let, for example, $j = 1$. If we can show that for every $\tau > 0$ there is a C (that may depend on L) such that for any $0 < \eta < 1$

$$\begin{aligned}&\int_0^{\eta x_1} \log \left| \frac{x - t + iL \, |I_1|}{x-t} \right| v(t) \, dt \\ &\quad \leqslant C\eta^{1-\tau} \int_0^{x_1} \log \left| \frac{x - t + iL \, |I_1|}{x-t} \right| v(t) \, dt, \qquad (4.23)\end{aligned}$$

then the preceding proof can be applied, for then first we choose an η for which the integral over $[0, \eta x_1]$ is small compared to the integral over $[0, x_1]$, and then apply that $v(t) = (1+o(1)) \, v(u)$ uniformly in $t, u \in [\eta x_{1,n}, x_{1,n}]$ (cf. the argument in Lemma 4) and then apply again (now with $v(t) \equiv 1$) that the integral over $[0, \eta x_1]$ is small compared to the one over $[0, x_1]$.

Thus, we have to prove (4.23). If $|x| \geqslant 2x_1$, then the logarithmic factor in both integrals is $\sim \log |x + iLx_1|/|x|$, and so in this case the claim follows from (4.5). If $2\eta x_1 \leqslant |x| \leqslant 2x_1$, then on the left hand side the integrand before $v(t)$ is at most $\log(2L/\eta)$, hence in this case the claim follows from (4.5) (applied to $\tau/2$ instead of τ) and from the fact that for $|x| \leqslant 2x_1$ the integrand before $v(t)$ on the right is at least

$$\log \frac{L\,|I_1|}{|x|+|t|} \geqslant \log L/3.$$

If, however, $|x| \leqslant 2\eta x_1$, then (by omitting the integrands just to indicate how we change the range of integration)

$$\int_0^{\eta x_1} \leqslant \int_{t \in I_1:\ |x-t| \leqslant 3\eta x_1},$$

therefore it is enough to show that

$$\int_{t \in I_1:\ |x-t| \leqslant 3\eta x_1} \log \left| \frac{x-t+iL\,|I_1|}{x-t} \right| v(t)\,dt$$
$$\leqslant C\eta^{1-\tau} \int_0^{x_1} \log \left| \frac{x-t+iL\,|I_1|}{x-t} \right| v(t)\,dt. \tag{4.24}$$

On the left we write the integral as the sum

$$\sum_{k=0}^{\infty} \int_{t \in I_1:\ 3\eta x_1/2^{k+1} \leqslant |x-t| \leqslant 3\eta x_1/2^k}.$$

Here in the kth integral

$$\log \left| \frac{x-t+iL\,|I_1|}{x-t} \right| \leqslant \log \frac{4L}{3\eta/2^k}.$$

Now we apply (4.6) to the (the one or two) interval(s) I of the kth integral, which is of length at most $3\eta x_1/2^k$. This way we obtain that the left hand side of (4.24) is not bigger than

$$2 \sum_{k=0}^{\infty} \left(\log \frac{4L}{3\eta/2^k} \right) C \left(\frac{3\eta}{2^k} \right)^{1-\tau} \int_0^{x_1} v(t)\,dt.$$

Here the last integral is clearly smaller than the integral on the right-hand side of (4.23), and the sum in front of this integral can be written as

$$2 \sum_{k=0}^{\infty} \left(\log \frac{4L2^k}{2\eta} \right) \left(\frac{3\eta}{4L2^k} \right)^{\tau} (4L)^{\tau}\, C \left(\frac{3\eta}{2^k} \right)^{1-2\tau} \leqslant C_L \eta^{1-2\tau},$$

because the function $(\log x)/x^{\tau}$ is bounded on $[1, \infty)$.

Here we have $1-2\tau$ in the exponent, but since τ was arbitrary, this does not matter and the proof is complete. ▮

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